

Lecture 10

Interior $C^{2,\alpha}$ estimate for Newtonian potential (continued)

Proposition 1 Consider $B_1 = B_R(x_0) \subset B_{2R}(x_0) = B_2$, $f \in C^\alpha(B_{2R})$, where $0 < \alpha < 1$. Let $\omega(x) = \int_{B_2} \Gamma(x-y)f(y)dy$, the Newtonian Potential of f in B_2 . Then $\omega \in C^{2,\alpha}(B_R(x_0))$ and we have estimate

$$\|D^2\omega\|_{0;B_R} + R^\alpha |D^2\omega|_{\alpha;B_R} \leq C(\|f\|_{0;B_{2R}} + R^\alpha |f|_{\alpha;B_{2R}}),$$

where $C = C(n, \alpha)$ is constant.

Proof:(continued)

$$\begin{aligned} (VI) &= (f(x) - f(\bar{x})) \int_{B_2 \setminus B_\delta(\xi)} D_{ij}\Gamma(x-y)dy \\ &\leq |f|_{C^\alpha(x)} |x - \bar{x}|^\alpha \left| \int_{\partial(B_2 \setminus B_\delta(\xi))} D_i\Gamma(x-y)\nu_j ds_y \right| \\ &\leq |f|_{C^\alpha(x)} \delta^\alpha \left(\left| \int_{\partial B_2} D_i\Gamma(x-y)\nu_j ds_y \right| + \left| \int_{\partial B_\delta(\xi)} D_i\Gamma(x-y)\nu_j ds_y \right| \right) \\ &\leq c|f|_{C^\alpha(x)} \delta^\alpha \left(\int_{\partial B_2} \frac{1}{|x-y|^{n-1}} ds_y + \int_{\partial B_\delta(\xi)} \frac{1}{|x-y|^{n-1}} ds_y \right) \quad \left(\frac{\delta}{2} \leq \frac{1}{2}|y-\xi| \leq |y-x| \right) \\ &\leq c|f|_{C^\alpha(x)} \left(\delta^\alpha \frac{1}{R^{n-1}} n\omega_n(2R)^{n-1} + \frac{1}{(\delta/2)^{n-1}} n\omega_n(\delta)^{n-1} \right) \\ &\leq c|f|_{C^\alpha(x)} \delta^\alpha. \end{aligned}$$

$$\begin{aligned} (V) &= \int_{B_2 \setminus B_\delta(\xi)} (D_{ij}\Gamma(\bar{x}-y) - D_{ij}\Gamma(x-y))(f(y) - f(\bar{x}))dy \\ &\leq \int_{B_2 \setminus B_\delta(\xi)} |DD_{ij}\Gamma(\hat{x}-y)| |x - \bar{x}| |f|_{C^\alpha(\bar{x})} |\bar{x} - y|^\alpha dy \\ &\leq c\delta |f|_{C^\alpha(\bar{x})} \int_{|y-\xi| \geq \delta} \frac{1}{|\hat{x}-y|^{n+1}} |\bar{x} - y|^\alpha dy \end{aligned}$$

Since

$$|\bar{x} - y| \leq |\bar{x} - \xi| + |\xi - y| \leq \frac{\delta}{2} + |\xi - y| \leq \frac{3}{2}|\xi - y|$$

and

$$\begin{aligned} |y - \xi| &\leq |y - \hat{x}| + |\hat{x} - \xi| \leq |y - \hat{x}| + \frac{\delta}{2} \leq |y - \hat{x}| + \frac{1}{2}|y - \xi| \\ \implies \frac{1}{2}|y - \xi| &\leq |y - \hat{x}|, \end{aligned}$$

We thus get

$$\begin{aligned}
(V) &\leq c\delta |f|_{C^\alpha(\bar{x})} \int_{|y-\xi| \geq \delta} \frac{dy}{|y-\xi|^{n+1-\alpha}} \leq c\delta |f|_{C^\alpha(\bar{x})} \int_{\delta}^{3R} \frac{1}{r^{n+1-\alpha}} r^{n-1} dr \\
&\leq c\delta |f|_{C^\alpha(\bar{x})} \int_{\delta}^{3R} r^{\alpha-2} dr \leq c\delta |f|_{C^\alpha(\bar{x})} \frac{1}{\alpha-1} ((3R)^{\alpha-1} - \delta^{\alpha-1}) \\
&\leq \frac{c}{1-\alpha} |f|_{C^\alpha(\bar{x})} \delta^\alpha.
\end{aligned}$$

Combine all the results, we have shown

$$|D_{ij}\omega(\bar{x}) - D_{ij}\omega(x)| \leq C\left(\frac{|f(x)|}{R^\alpha} + |f|_{C^\alpha(x)} + |f|_{C^\alpha(\bar{x})}\right)|x - \bar{x}|^\alpha,$$

thus

$$R^\alpha \frac{|D_{ij}\omega(\bar{x}) - D_{ij}\omega(x)|}{|x - \bar{x}|^\alpha} \leq C(|f(x)| + R^\alpha |f|_{C^\alpha(x)} + R^\alpha |f|_{C^\alpha(\bar{x})}),$$

i.e.

$$R^\alpha |D^2\omega|_{\alpha; B_R} \leq C(\|f\|_{C^0; B_{2R}} + R^\alpha |f|_{C^\alpha(B_{2R})}).$$

Since we have already known that

$$|D^2\omega|_0 \leq c(\|f\|_{C^0} + R^\alpha |f|_\alpha),$$

we finally get

$$\|D^2\omega\|_{0; B_R} + R^\alpha |D^2\omega|_{\alpha; B_R} \leq C(\|f\|_{0; B_{2R}} + R^\alpha |f|_{\alpha; B_{2R}}). \quad \blacksquare$$

Exercise:

- 1) Find a continuous function f s.t. $\Delta u = f$ does not have a C^2 solution.
- 2) Find $g \in C^1$ and $\Delta u = g$ but u is not $C^{2,1}$.

Interior $C^{2,\alpha}$ estimates for Poisson's equation.

Application: $u \in C^2(B_{2R}(x_0)), \Delta u = f, f \in C^\alpha(B_{2R}(x_0)).$

Theorem 1

$$\|u\|_{C^{2,\alpha}(B_R)} \leq \frac{C}{R^\alpha} (\|u\|_{C^0(B_{2R})} + \|f\|_{C^\alpha(B_{2R})}).$$

Proof: Since $\Delta(u - Nf) = 0$, where Nf is the Newtonian potential of f , thus from the $C^{2,\alpha}$ estimate of Nf we can get the $C^{2,\alpha}$ of u . ■

Theorem 2 Let $u \in C^2(\Omega)$, $\Delta u = f$, $f \in C^\alpha(\Omega)$, then for any $\Omega' \subset \subset \Omega$, we have

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{C^\alpha(\Omega)}).$$

Proof: Take $x \in \Omega'$, choose R s.t. $B_{2R} \subset \Omega$, then

$$\begin{aligned} |u|_{C^{2,\alpha}(x)} &\leq |u|_{C^{2,\alpha}(B_R(x))} \leq \frac{C}{R^\alpha}(\|u\|_{C^0;B_{2R}} + |f|_{C^\alpha(B_{2R})}) \\ &\leq \frac{C}{R^\alpha}(\|u\|_{C^0;\Omega} + |f|_{C^\alpha(\Omega)}). \end{aligned}$$

Taking superior over all x and using previous estimate, we get

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{C^\alpha(\Omega)}). \quad \blacksquare$$

Boundary estimate on Newtonian potential: $C^{2,\alpha}$ estimate up to the boundary for domain with flat boundary portion.

Suppose $B_R \subset B_{2R}$, with flat boundary portion $B_1^+ \subset B_2^+$.

Lemma 1 Let $f \in C^\alpha(\overline{B_2^+})$, $\omega(x) = \int_{B_2^+} \Gamma(x-y)f(y)dy$ be the Newtonian potential of f in B_2^+ . Then $\omega \in C^{2,\alpha}(\overline{B_1^+})$ and

$$|D^2\omega|_{0;B_1} + R^\alpha |D^2\omega|_{\alpha;B_1^+} \leq C(|f|_{0;B_2} + R^\alpha |f|_{\alpha;B_2^+}).$$

Proof: Examine the proof from last time. From C^2 estimate have

$$D_{ij}\omega(x) = \int_{B_2^+} D_{ij}\Gamma(x-y)(f(y) - f(x))dy - f(x) \int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)ds_y.$$

(Note $\int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)ds_y = \int_{\partial B_2^+} D_j\Gamma(x-y)\nu_i(y)ds_y$.)

So if either $i \neq n$ or $j \neq n$, the integral on the lower boundary portion of B_2^+ vanishes. (Since $\nu = (0, 0, \dots, -1)$.)

If $i = j = n$, then

$$\begin{aligned} D_{nn}\omega &= f(x) \int_{\partial B_2^+} (D_n\Gamma(x-y) - D_n\Gamma(\bar{x}-y))(-1)d\sigma \\ &\leq |f(x)| \int_{\partial B_2^+} |DD_n\Gamma(\hat{x}-y)||x-\bar{x}|d\sigma \\ &\leq |f(x)|\delta \int_{\partial B_2^+} \frac{1}{|\hat{x}-y|^n}d\sigma \\ &\leq |f(x)|\delta \frac{1}{R^n} n\omega R^{n-1} \\ &\leq c|f(x)|\delta^\alpha. \end{aligned}$$

Since we know $\Delta\omega = f$, thus $\omega_{nn} = f - \omega_{11} - \omega_{22} - \dots$, so we see that $\omega_{nn} \in C^\alpha$, and we can get estimate for ω_{nn} from estimates for $\omega_{ii}, i < n$. $\blacksquare$